Lecture 15

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ROLLE'S THM

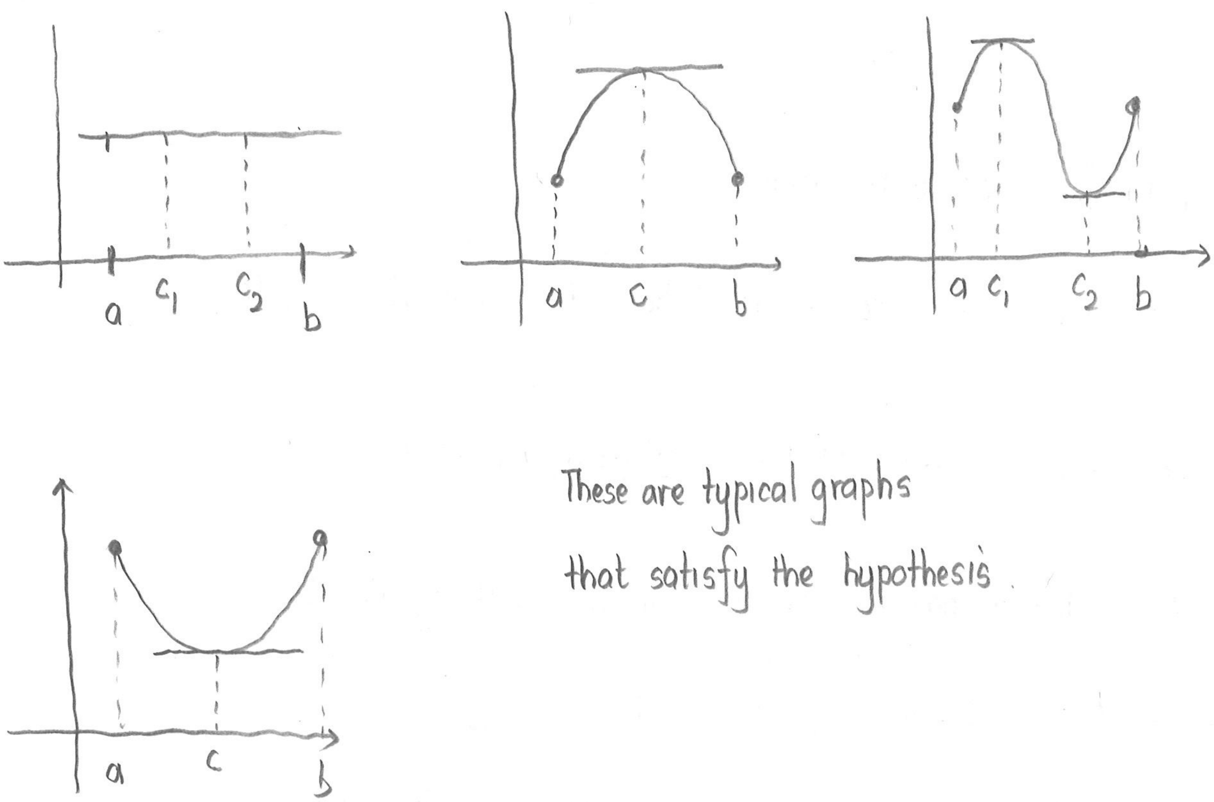
f be a function that satisfies the following 3 hypothesis :

1) f is continuous on the closed interval $[a,b]$.

2) f is differentiable on the open interval (a,b).

3) $f(a) = f(b)$

Then there is a number c in (a,b) such that $f'(c) = 0$.



These are typical graphs that satisfy the hypothesis .

Pf

Case 1 $f(x) = K$, a constant

Then $f'(x) = 0$, so any number c between a & b works.

Case 2 $f(x) > f(a)$ for some x in (a,b)

By Extreme Value Thm (which we can apply because f is continuous on $[a,b]$),

so f has a maximum value in $[a,b]$. Since $f(a) = f(b)$, it must

attain maximum value at a number c in (a,b) since can't happen on endpoints. The f has a local

max at c, and since f is differentiable at c. Then by Fermat's Thm,

$f'(c) = 0$.

Case 3 $f(x) < f(a)$ for some x in (a,b)

Similar reasoning as above works. We see that f has a local min at c & $f'(c) = 0$.

Example Prove that eqn $x^3 + x - 1 = 0$ has exactly one real root.

Soln STEP 1 SHOW that it has at least one real root. (Using IVT)

Let $f(x) = x^3 + x - 1$. $f(0) = -1$, $f(1) = 1$

Since f is polynomial, it is continuous, and by Intermediate value Thm,

there is a number c between 0 and 1 such that $f(c) = 0$.

Thus f has a root.

STEP 2 We would like to show that the equation has no other real root

Use Rolle's Thm and argue by contradiction.

Spse f has two roots a and b.

Then $f(a) = 0 = f(b)$, and since f is a polynomial, it is differentiable on (a,b) and continuous on $[a,b]$.

Then by Rolle's Thm there is a number c between a and b such that

$f'(c) = 0$.

But $f'(x) = 3x^2 + 1 \geqslant 1$, so $f'(x)$ can never be 0.

Hence f cannot have two roots.

Therefore, f has exactly 1 root.

ROLLE'S THM IS important in proving the important Thm, called Mean Value Thm (M.V.T).

THE MEAN VALUE THM

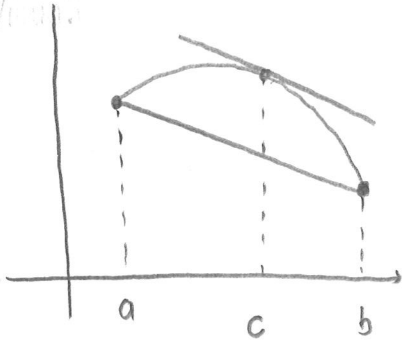
Let f be a function that satisfies the following hypothesis:

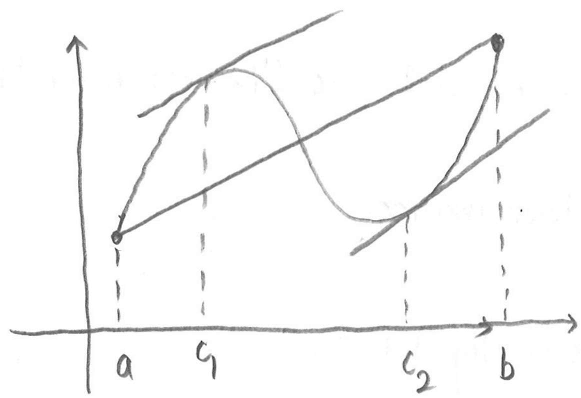
1. f is continuous on the closed interval $[a,b]$.
2. f is differentiable on the open interval (a,b).

Then there is a number c in (a,b) s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{i.e.} \quad \boxed{f(b)-f(a) = f'(c)(b-a).}$$

<u>Geometrically</u>, if $A = (a, f(a))$, $B = (b, f(b))$,
then the MVT says that there is at least one point $(c, f(c)) = P$ ($\to$ could be more)
where the tangent line is parallel to the secant line (i.e slope of tangent line is equal to the slope of the secant line)





<u>Remark</u> If $f(a) = f(b)$, then we get the same statement as Rolle's Thm.

<u>Pf of MVT</u> The basic Idea is we apply Rolle's Thm to

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a) \text{ but}$$

first we need to verify it satisfies the three hypothesis of the Rolle's Thm.

1. The function h is continuous on $[a, b]$ because it is a sum of a continuous function and a polynomial.

2. The function h is differentiable on (a, b) because both f and the polynomial differentiable. Actually,

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}$$

[Since $f(a)$ and $\frac{f(b)-f(a)}{b-a}$ are constants]

3. $$h(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0$$

$$h(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a)$$
$$= f(b) - f(a) - [f(b) - f(a)] = 0$$

So, $h(a) = h(b)$.

Then by Rolle's Thm, there is a number c in (a, b) s.t $h'(c) = 0$.

Therefore, $0 = h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$ and hence

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Ex Spse $f(1) = 3$ and $f'(x) \leq 11$ for all values of x.

How large can be $f(4)$ be ?

Soln We are given that f is differentiable (and hence continuous) everywhere. Namely, we can apply MVT on the interval $[1,4]$.

Then, there exists a number s.t

$$f(4) - f(1) = f'(c)(4-1)$$

$$\Rightarrow f(4) = 3f'(c) + f(1) = 3f'(c) + 3$$

Since $f'(x) \leq 11$ for all x, $f'(c) \leq 11 \Rightarrow 3f'(c) \leq 33$

$$f(4) \leq 33 + 3 = 36$$

So largest possible value of $f(4)$ is 36.

Thm If $f'(x) = 0$ for all x in the interval (a,b), then f is constant on (a,b).

Pf Let x_1 and x_2 be any two numbers in (a,b) with $x_1 < x_2$. Since f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$.

Then by using MVT to f on the interval $[x_1, x_2]$, we get a number c such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$

Since $f'(x) = 0$ for all x in (a, b), $f'(c) = 0$, and so

$$f(x_2) - f(x_1) = 0 \Rightarrow f(x_1) = f(x_2)$$

So f has the same value at any two numbers x_1 and x_2 in (a,b).

So f is constant on (a,b).

Corollary If $f'(x) = g'(x)$ for all x in an interval (a,b), then $f-g$ is constant on (a,b); i.e. $f(x) = g(x) + c$ where c is a constant.

Pf let $F(x) = f(x) - g(x)$

Then $F'(x) = f'(x) - g'(x) = 0$ for all x in (a,b)

So $F(x)$ is constant i.e $f-g$ is constant.